Ejemplos

Determina las singularidades y clasificar.

(A) $f(z) = \dfrac{(\cos z - 1)\, e^{\frac{1}{z+1}}}{z}$

Sing: $z_0 = 0$
$z_1 = -1$.

si existen

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{(\cos z - 1)}{z} \cdot e^{1/z+1} = \lim_{z\to 0} \frac{(\cos z-1)}{z} \cdot \lim_{z\to 0} e^{1/z+1}$$

$$= \lim \frac{-\operatorname{sen} z}{1} \cdot \lim e^{1/z+1} = 0 \cdot e = 0$$

$\Rightarrow$ $z_0 = 0$ es evitable. $\operatorname{Res}(f, 0) = 0$

$z_1 = -1$ es esencial:

Taylor

$\varphi(z) = \dfrac{\cos z - 1}{z}$ es holomorfa en -1: $\varphi(z) = \displaystyle\sum_{k=0}^{\infty} a_k (z+1)^k$
$= a_0 + a_1(z+1) + a_2(z-1)^2 + \dots$

$$e^{\frac{1}{z+1}} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(z+1)^k} = 1 + \frac{1}{z+1} + \frac{1}{2!(z+1)^2} + \dots$$

$$\Rightarrow f(z) = \varphi(z) e^{\frac{1}{z+1}} = \left(a_0 + a_1(z+1) + a_2(z-1)^2 + \dots\right)\left(1 + \frac{1}{z+1} + \frac{1}{2!(z+1)^2} + \dots\right)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (a_{k-j} b_j)(z+1)^k \qquad b_j = \begin{cases} 0 & j \geq 1 \\ \frac{1}{(-j)!} & j \leq 0 \end{cases}$$

Es esencial ~~pq~~ ya que:

$$\lim_{z\to -1} \frac{(\cos z - 1)}{z} e^{1/z+1} = \lim_{w\to 0} \frac{\cos(w-1)-1}{w-1} \cdot e^{1/w} \not\exists$$

$w = z+1$

$$\operatorname{Res}(f, -1) = \text{coef de } \frac{1}{z+1} \text{ en } \cancel{\text{la}} \text{ DSL} = a_0 + \frac{a_1}{2!} + \frac{a_2}{3!} + \dots$$

(B) $f(z) = \dfrac{z}{\cos z - 1}$

Sing: $z : \cos z = 1$

$$z_k = 2k\pi \quad , k \in \mathbb{Z}$$

$z_0 = 0$ → es cero orden 1 del numerador y cero orden 2 del denom.

$g(z) = \cos z - 1 \quad \rightarrow g(0) = 0$

$g'(0) = -\operatorname{sen} z|_0 = 0$

$g''(0) = -\cos z|_0 = -1 \neq 0$

⇒ $z_0 = 0$ es polo de orden 1 de f.

Otra forma: $\lim_{z \to 0} f(z) = \lim_{z \to 0} \dfrac{z}{\cos z - 1} \overset{L'H}{=} \lim \dfrac{1}{-\operatorname{sen} z} = \infty$

$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \dfrac{z^2}{\cos z - 1} = \lim \dfrac{2z}{-\operatorname{sen} z} = \lim \dfrac{2}{-\cos z} \neq 0$

⇒ polo orden 1.

$\operatorname{Res}(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \dfrac{2}{-\cos z} = -2$

$z_k = 2k\pi$, $k \neq 0$ $k \in \mathbb{Z}$

↳ cero orden 2 del denominador, no anula numerador.

$g(z) = \cos z - 1 \quad \rightarrow g(2k\pi) = 0$

$g'(2k\pi) = -\operatorname{sen}(2k\pi) = 0$

$g''(2k\pi) = -\cos(2k\pi) = -1 \neq 0$.

⇒ z_k polo orden 2 de f.

Otra forma: $\lim_{z \to z_k} f(z) = \infty$

$\lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} \dfrac{(z - 2k\pi) \cdot z}{\cos z - 1} \overset{L'H}{=} \lim_{z \to z_k} \dfrac{2z - 2k\pi}{-\operatorname{sen} z} = \infty$

$$\lim_{z\to z_k} (z-z_k)^2 f(z) = \lim_{z\to z_k} \frac{(z-2k\pi)^2 z}{\cos z - 1} \overset{L'H}{=} \lim_{z\to z_k} \frac{2z(z-2k\pi)+(z-2k\pi)^2}{-\operatorname{sen} z}$$

$$= \lim_{z\to z_k} \frac{(z-2k\pi)(2z+z-2k\pi)}{-\operatorname{sen} z} = \lim_{z\to z_k} \frac{4z-4k\pi}{-\cos z} = -4k\pi \neq 0$$

$\Rightarrow$ es polo doble.

ⓒ $f(z) = \dfrac{z^2-1}{(z^2+1)^2}$

Sing: $\left.\begin{array}{l} z_0 = i \\ z_1 = -i \end{array}\right\}$ ambos polos dobles:

$$\lim_{z\to i} f(z) = \lim_{z\to i} \frac{z^2-1}{(z^2+1)^2} = \infty$$

$$\lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \frac{(z-i)(z^2-1)}{(z-i)^2(z+i)^2} = \infty$$

$$\lim_{z\to i} (z-i)^2 f(z) = \lim_{z\to i} \frac{\cancel{(z-i)^2}(z^2-1)}{\cancel{(z-i)^2}(z+i)^2} = \frac{-2}{(2i)^2} \neq 0$$

$\Rightarrow$ i polo doble.

Similarmente con $z_1 = -i$

$$\operatorname{Res}(f, i) = \lim_{z\to i} \left[(z-i)^2 f(z)\right]' = \lim_{z\to i} \left(\frac{z^2-1}{(z+i)^2}\right)' =$$

$$\lim_{z\to i} \frac{2z(z+i)^2 - (z^2-1)\,2(z+i)}{(z+i)^4} = \frac{2i(2i)^2 - (-2).2.2i}{(2i)^4} =$$

$$= \frac{2i(-4)+8i}{(-4)^2} = 0$$

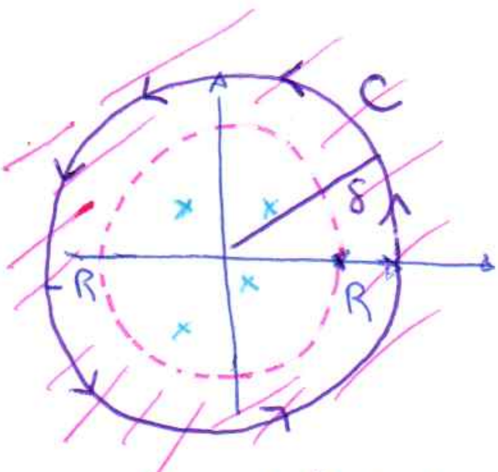
Singularidad en infinito

f es holomorfa en $|z|>R \iff \infty$ es una singularidad aislada

DSL en $\{z\in\mathbb{C} : |z|>R\}$:

$$f(z) = \underbrace{\sum_{k=-\infty}^{\infty} c_k z^k}_{} = \dots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2$$

"Desarrollo en serie en torno al infinito".
"en una vecindad de infinito"



$\boxed{-c_{-1} : \text{residuo de } f \text{ en } \infty}$

Si C: circunf de radio $\delta > R$:

$$\int_C f(z)\,dz = \int_C \sum_{-\infty}^{\infty} c_k z^k\,dz =$$

$$= \sum_{-\infty}^{\infty} c_k \underbrace{\int_C z^k\,dz}_{=\begin{cases} 0 & k\neq -1 \\ 2\pi i & k=-1\end{cases}} = 2\pi i\, c_{-1} = -2\pi i \operatorname{Res}(f,\infty)$$

*: sing de f, todas en $|z|<R$

¿Cómo calcular $\operatorname{Res}(f,\infty)$? → 1 opción: con el DSL en ∞.

Otra opción:

$$\int_C f(z)\,dz = \int_\gamma \underbrace{f\left(\tfrac{1}{w}\right)\left(\tfrac{-1}{w^2}\right)}_{\text{holo en } \gamma \text{ y } RI(\gamma)\text{, excepto } w=0} dw = -2\pi i \underbrace{\operatorname{Res}\left(\tfrac{-1}{w^2} f\left(\tfrac{1}{w}\right), 0\right)}_{\operatorname{Res}(f(z),\infty)}$$

porque γ está recorrida negativamente

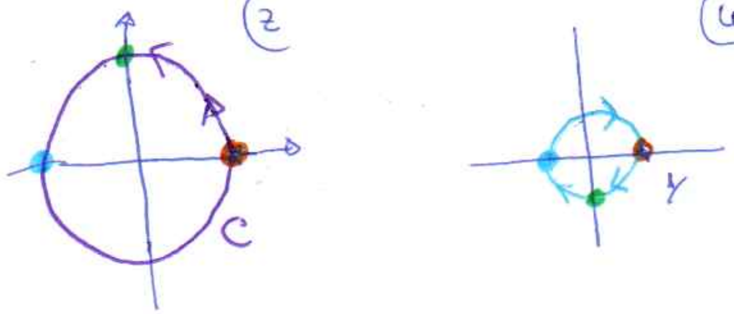
$w = \frac{1}{z}$

$dw = \frac{-1}{z^2}dz$

γ: circ $|z| = \frac{1}{\delta}$

recorrida negativamente

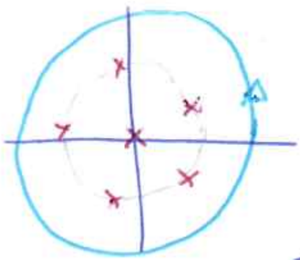
$f(z)$ holo si $|z|>R$
$\Rightarrow f\left(\frac{1}{w}\right)$ holo si $\left|\frac{1}{w}\right|>R$,
o sea: $0<|w|<\frac{1}{R}$



$$\boxed{\operatorname{Res}(f(z),\infty) = \operatorname{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right)}$$

Ejemplo calcular $\displaystyle\int_C \frac{e^{1/z}}{z^5-2-i}\,dz \qquad C: |z|=20$

$f(z) = \dfrac{e^{1/z}}{z^5-2-i}$ → singularidades: $z=0$,
z tales que $z^5 = 2+i$ (5 soluciones)



todas las sing. satisfacen $|z|<20$

$\Rightarrow f$ es holo si $|z|\geq 20$

$$\Rightarrow \int_C \frac{e^{1/z}}{z^5-2-i}\,dz = -2\pi i \operatorname{Res}(f,\infty)$$

$$\operatorname{Res}(f,\infty) = \operatorname{Res}\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = \operatorname{Res}\left(-\frac{1}{z^2}\,\frac{e^z}{\frac{1}{z^5}-2-i}, 0\right)$$

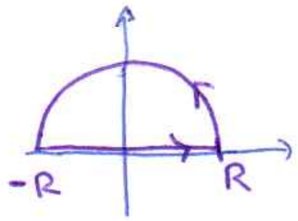
$$g(z) = -\frac{1}{z^2} f\left(\frac{1}{z}\right) = -\frac{1}{z^2}\,\frac{e^z}{\frac{1}{z^5}-2-i} \underset{\text{si } z\neq 0}{=} \frac{-e^z\cdot z^3}{1-(2+i)z^5}$$

$\Rightarrow 0$ es sing evitable de g $\Rightarrow \operatorname{Res}(g,0) = \operatorname{Res}\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 0$

$$\Rightarrow \int_C \frac{e^{1/z}}{z^5-2-i}\,dz = 0$$

Show de ejemplos

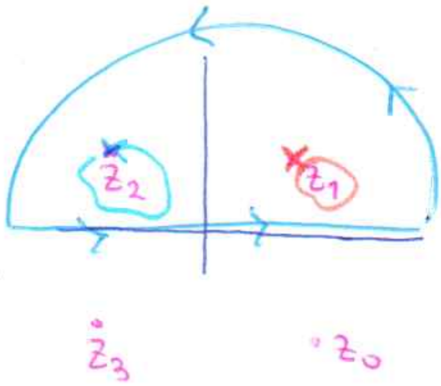
① $\displaystyle\int_C \frac{1}{z^4+1}\,dz$ $\quad C:$



$R>1$

Sing: $z:\ z^4=-1 \longrightarrow e^{i4\theta} = e^{-\pi i} \longrightarrow \theta = -\frac{\pi}{4} + \frac{2k\pi}{4} = -\frac{\pi}{4} + \frac{k\pi}{2}$ $\quad z = e^{i\theta}$ $\quad k \in \mathbb{Z}$

$z_0 = e^{-\pi/4\, i} \qquad z_1 = e^{\pi/4\, i} \qquad z_2 = e^{3/4\,\pi i} \qquad z_3 = e^{5/4\,\pi i}$



$$\int_C \frac{1}{z^4+1}\,dz = 2\pi i\left(\operatorname{Res}\left(\frac{1}{z^4+1}, z_1\right) + \operatorname{Res}\left(\frac{1}{z^4+1}, z_2\right)\right)$$

Calculamos residuos:

z_k son polos simples (son ceros de orden 1 del denominador, no anulan numerador)

$$\operatorname{Res}\left(\frac{1}{z^4+1}, z_k\right) = \lim_{z\to z_k} (z-z_k)\frac{1}{z^4+1} \underset{L'H}{=} \lim_{z\to z_k} \frac{1}{4z^3} = \frac{1}{4z_k^3}$$

$$\operatorname{Res}\left(\frac{1}{z^4+1}; z_1\right) = \frac{1}{4e^{3/4\,\pi i}} = \frac{e^{-\frac{3\pi}{4}i}}{4}$$

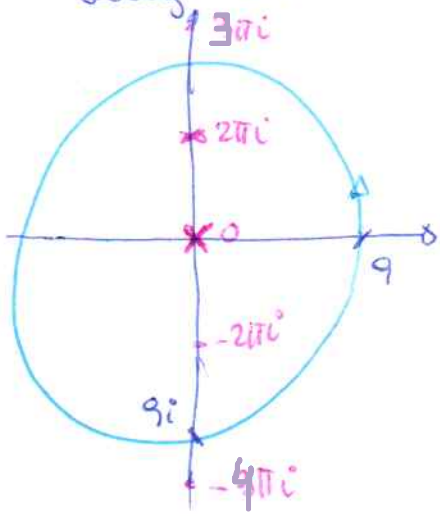
$$\operatorname{Res}\left(\frac{1}{z^4+1}, z_2\right) = \frac{1}{4e^{\frac{9}{4}\pi i}} = \frac{e^{-\frac{9}{4}\pi i}}{4} = \frac{e^{-\frac{\pi i}{4}}}{4}$$

$$\int_C \frac{1}{z^4+1}\,dz = 2\pi i\left(\frac{1}{4}e^{-\frac{3\pi}{4}i} + \frac{1}{4}e^{-\frac{\pi}{4}i}\right) = \frac{\pi i}{2}\left(\cos\left(-\tfrac{3\pi}{4}\right) + i\,\text{sen}\left(-\tfrac{3\pi}{4}\right) + \cos\left(-\tfrac{\pi}{4}\right) + i\,\text{sen}\left(-\tfrac{\pi}{4}\right)\right)$$

$$= \frac{\pi i}{2}\left(\cancel{-\frac{\sqrt2}{2}} - i\frac{\sqrt2}{2} + \cancel{\frac{\sqrt2}{2}} - i\frac{\sqrt2}{2}\right) = \pi\frac{\sqrt2}{2}$$

② $\int_C \frac{1}{e^z-1}\,dz \qquad C: |z|=9$

Sing: $z: e^z=1 \longrightarrow z=2k\pi i\ ,\ k\in\mathbb{Z}.$



Sing en $Rl(C)$: $\underbrace{0}_{z_0}, \underbrace{2\pi i}_{z_1}, \underbrace{-2\pi i}_{z_2}$.

Tipo sing?

$\lim_{z\to z_k} \frac{1}{e^z-1} = \infty \Rightarrow$ no son evitables.

$\lim_{z\to z_k} (z-z_k)\frac{1}{e^z-1} \overset{L'H}{=} \lim_{z\to z_k} \frac{1}{e^z} = \frac{1}{e^{z_k}} \neq 0$

$\Rightarrow$ son polos orden 1.

$Res(f, z_k) = \lim_{z\to z_k} (z-z_k)\frac{1}{e^z-1} = \frac{1}{\underbrace{e^{z_k}}_{=1}} = 1$

$\Rightarrow \int_C \frac{1}{e^z-1}\,dz = 2\pi i\left(Res(f,z_0)+Res(f,z_1)+Res(f,z_2)\right) =$

$= 2\pi i \cdot 3 = 6\pi i$

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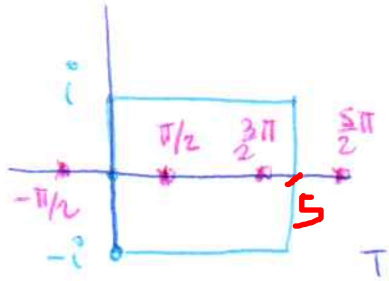
③ $\int_C \tan z\,dz \qquad C$: rectángulo de vértices $-i, -i+5, i+5, i$

Sing de $f(z)=\tan z = \frac{\operatorname{sen} z}{\cos z}$

$\hookrightarrow z: \cos z = 0 \Rightarrow z_k = \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$

$z_0 = \pi/2$ y $z_1 = \frac{3}{2}\pi$ están en interior de C.



Tipo?

$\lim_{z\to z_k} f(z) = \lim_{z\to z_k}\frac{\operatorname{sen} z}{\cos z} = \infty \rightarrow$ no es evitable

$\int = 2\pi i \cdot (-2)$

$\lim_{z\to z_k}(z-z_k)f(z) = \lim_{z\to z_k}(z-z_k)\frac{\operatorname{sen} z}{\cos z} =$

$= \lim \frac{\operatorname{sen} z + (z-z_k)\cos z}{-\operatorname{sen} z} = -1$

falta lógica

$$\operatorname{Res}\left(\frac{e^z-1}{\operatorname{sen}^3 z}, 0\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2(e^z-1)}{\operatorname{sen}^3 z}\right) =$$

$$= \lim_{z\to 0} \frac{\left(\left(2z(e^z-1)+z^2e^z\right)\operatorname{sen}^3 z - z^2(e^z-1)\,3\operatorname{sen}^2 z\cos z\right)}{\operatorname{sen}^6 z}$$

$$= \lim_{z\to 0} \frac{e^z(2z+z^2)-2z}{\operatorname{sen}^3 z} - \frac{3z^2(e^z-1)\cos z}{\operatorname{sen}^4 z} =$$

$$= \lim_{z\to 0} \frac{e^z\cdot z(z+2)\operatorname{sen} z - 2z\operatorname{sen} z - 3z^2(e^z-1)\cos z}{\operatorname{sen}^4 z} =$$

$$= \lim_{z\to 0} \frac{z\left[e^z\left((z+2)\operatorname{sen} z - 3z\cos z\right) + 3z\cos z - 2\operatorname{sen} z\right]}{\operatorname{sen}^4 z} = \frac{1}{2}$$

$$\Rightarrow \int_C \frac{e^z-1}{\operatorname{sen}^3 z}\,dz = 2\pi i\cdot\frac{1}{2} = \pi i$$

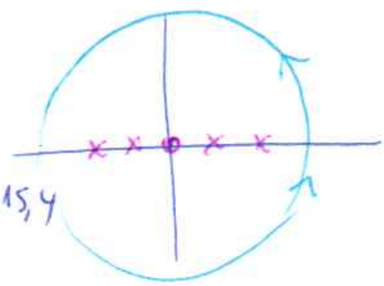
(5)

$$\int_C \frac{1}{z\operatorname{sen}\left(\frac{1}{z}\right)}\,dz \qquad C:\ |z| = 15{,}4$$

Sing: $z_0 = 0$ y z tales que $\operatorname{sen}\left(\frac{1}{z}\right) = 0 \to \frac{1}{z} = k\pi \to z_k = \frac{1}{k\pi},\ k\in\mathbb{Z}$

$z_0 = 0$: sing. no aislada

todas las sing satisfacen $|z_k| \le \frac{1}{\pi} < 15{,}4$



$\Rightarrow$ todas en $\operatorname{Int}(C) \Rightarrow f(z) = \frac{1}{z\operatorname{sen}\left(\frac{1}{z}\right)}$ es holo en $|z| \geq 15{,}4$

$$\Rightarrow \int_C \frac{1}{z\operatorname{sen}\left(\frac{1}{z}\right)}\,dz = -2\pi i\operatorname{Res}(f,\infty)$$

$$\operatorname{Res}(f,\infty) = \operatorname{Res}\left(-\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right)$$

$$g(z) = -\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{\frac{1}{z}\operatorname{sen}(z)}\cdot\left(-\frac{1}{z^2}\right) = \frac{-1}{z\operatorname{sen} z}.$$

$\to$ ¿qué tipo sing tiene en 0?

$z_0=0$ es cero de orden 2 del denominador:

$h(z) = z \operatorname{sen}(z)$ $\quad h(0)=0$

$h'(z) = \operatorname{sen} z + z \cos z$ $\quad h'(0)=0$

$h''(z) = \cos z + \cos z + z \operatorname{sen} z$ $\quad h''(0) = 2 \neq 0$

$z_0=0$ no anula numerador $\Rightarrow$

$z_0=0$ es polo orden 2 de $g(z) = \dfrac{-1}{z \operatorname{sen} z}$.

$$\operatorname{Res}(g,0) = \lim_{z\to 0} \frac{d}{dz}\left(z^2\left(\frac{-1}{z \operatorname{sen} z}\right)\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{-z}{\operatorname{sen} z}\right) =$$

$$= \lim_{z\to 0} \frac{-\operatorname{sen} z + z\cos z}{\operatorname{sen}^2 z} \overset{L'H}{=} \lim_{z\to 0} \frac{-\cancel{\cos z} + \cancel{\cos z} - z \operatorname{sen} z}{2 \operatorname{sen} z \cos z} =$$

$$= \lim_{z\to 0} \frac{-z\,\cancel{\operatorname{sen} z}}{2\,\cancel{\operatorname{sen} z}\cos z} = 0$$

$$\Rightarrow \int_C \frac{1}{z \operatorname{sen}\left(\frac{1}{z}\right)}\,dz = -2\pi i \operatorname{Res}(f,\infty) = -2\pi i \operatorname{Res}(g,0) = 0.$$

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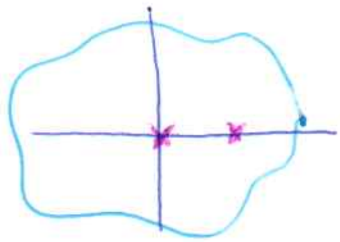
(6) $\displaystyle\int_C \frac{e^{z-3}}{z^3-z^2}\,dz$ $\qquad$ C:



Sing: $z_0=0$ y $z_1=1$.

$f(z) = \dfrac{e^{z-3}}{z^2(z-1)}$ $\longrightarrow$ $\left.\begin{array}{l}0 \text{ es polo doble} \\ 1 \text{ es polo simple.}\end{array}\right\}$ ambos interiores

$$\operatorname{Res}(f,0) = \lim_{z\to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{e^{z-3}}{z-1}\right) = \lim_{z\to 0} \frac{e^{z-3}(z-1) - e^{z-3}\cdot 1}{(z-1)^2}$$

$$= e^{-3}(-2)$$

$$\operatorname{Res}(f,1) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{e^{z-3}}{z^2} = \frac{e^{-2}}{1} = e^{-2}$$

$$\int_C \frac{e^{z-3}}{z^3-z^2}\,dz = 2\pi i\left(-2e^{-3} + e^{-2}\right)$$